Heterogeneity and Incompressibility in the Evolution of Elastic Wires

Elastic wires are mathematical curves composed of matter. They are used to model approximately one dimensional elastic objects like plant stems, polymers, marine cables or hair. The elastic energy of a sufficiently smooth regular curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ describing an elastic wire is defined as

$$\mathcal{E}(\gamma) = \frac{1}{2}\int_\gamma |\kappa|^2 \mathrm{d}s. \tag{1}$$

Here, $\kappa = \partial_s^2\gamma$ is the curvature of γ, $\partial_s = |\partial_x\gamma|^{-1}\partial_x$, and $\mathrm{d}s = |\partial_x\gamma|\mathrm{d}x$ is the arclength element. In the last decades, several authors have studied the L^2-gradient flow of the elastic energy in different variants. We briefly summarize their results and then focus on two new variants.

First, we consider elastic wires with a heterogeneity described by a density function. We define a generalization of (1), which depends on material parameters, captures the interplay between curvature and density effects and resembles the Canham–Helfrich functional. Describing the closed planar curve by its inclination angle, the L^2-gradient flow of this energy is a nonlocal coupled parabolic system of second order. We shortly discuss local well-posedness, global existence and convergence. Then, we show that the (non)preservation of quantities such as convexity as well as the asymptotic behavior of the system depend delicately on the choice of material parameters.

Second, we study the evolution of elastic wires under the assumption of incompressiblity and derive a gradient flow of (1) which preserves the enclosed area of the evolving planar curves. Contrary to an earlier approach using Lagrange multipliers, we give priority to the locality of the evolution equation, accepting it being of sixth order. We prove a global existence result and, by penalizing the length, we show convergence to an area constrained critical point of the elastic energy.